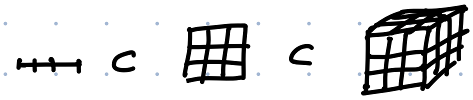
Lecture 27



Recall we have two descriptions of $H^*(G/B,\mathbb{R})$.

① Borel description: $H^*(G/B,\mathbb{R}) \cong \mathcal{R} := S(\mathfrak{t}^*)/\langle S(\mathfrak{t}^*)^W_+ \rangle$ ideal gen by. RING ISO.

where $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ is a Cartan and $\mathfrak{h} \cong \mathfrak{a} \oplus \mathfrak{t}$ as $\mathbb{R}$ Lie alg.
Roots$|_{\mathfrak{a}}$ real, Roots$|_{\mathfrak{t}}$ imag.

Or $\mathfrak{t} = \mathrm{Lie}(T)$, $T =$ maximal $(S^1)^r$ subgroup of $K =$ cpt real form.

Recall $S(V) = \bigoplus_{i \geq 0} V^{\otimes i} \cong \mathbb{R}[x_1,\dots,x_n]$ if $x_1 \dots x_n$ basis of V^*

$S(V)^W =$ W-invts. $S(V)^W_+ =$ positive degree invts.

② Schubert description $H_*(G/B,\mathbb{R}) \cong$ vec space$/\mathbb{R}$ w/ basis $\{X_w \mid w \in W\}$ and grading 2ℓ

$\ell : W \to \mathbb{Z}^{\geq 0}$ min length as word in $\{s_\alpha \mid \alpha \in \Delta\}$.

And $H^*(G/B,\mathbb{R})$ is the <u>dual</u> of this. (functions $W \to \mathbb{R}$)

$\boxed{\text{Basic question}}$ Describe the isomorphism between these two models of $H^*(G/B,\mathbb{R})$.

For example, one natural way to answer this question would be to give explicit elements of $\mathcal{R}$ that form a dual basis to the homology basis $\{X_w\}$.

Since $\mathcal{R}$ is a quotient of a polynomial ring, one could do this by finding elements $f_w \in S(\mathfrak{t}^*)$ such that
$\langle \alpha(f_w), X_{w'} \rangle = \delta_{ww'}$ where $\alpha : S(\mathfrak{t}^*) \twoheadrightarrow \mathcal{R}$ is the quotient
and $\langle , \rangle$ is the hom/cohom duality.

But such f_w will not be unique, of course.

Kostant (1963) and Bernstein-Gelfand-Gelfand (1973) gave answers. We'll follow the BGG approach which introduced a number of other ideas that turned out to have independent applications.

The orig paper is BGG, "Schubert cells and cohomology of the spaces G/P".

They split the problem into two steps.

① Duality formula. (or "integration formula" per BGG)

Given a polynomial $f \in S(\mathfrak{t}^*)$ and element $w \in W$, they give an algorithm to compute $\langle \underbrace{\alpha(f)}_{H^*}, \underbrace{X_w}_{H_*} \rangle$

② Dual basis

Using ①, they identify polynomials $f_w \in S(\mathfrak{t}^*)$ s.t. $\{\underbrace{\alpha(f_w)}_{H^*}\ w \in W\}$ is dual to $\{\underbrace{X_w}_{H_*} \mid w \in W\}$

<u>Now, let's get more precise.</u> First of all, lets treat elements of $S(\mathfrak{t}^*)$ as actual polynomials. Identify $\mathfrak{h} \overset{*}{\cong} \mathbb{C}^r$ in such a way that $\mathfrak{a}$ corresponds to $\mathbb{R}^r$ and $\mathfrak{t}$ to $i\mathbb{R}^r$.

Let f be a polynomial function on $\mathfrak{h}$ w/ <u>real coefs</u> (rel to $*$). So $f: \mathbb{C}^n \to \mathbb{C}$.

Then the map $\mathfrak{t} \to \mathbb{R}$ given by $\theta \longmapsto f(-i\theta)$ is an element of $S(\mathfrak{t}^*)$ (poly real-valued fn) and every elt arises uniquely.

Using this, we identify $S(\mathfrak{t}^*)$ with P = poly on $\underset{(\mathfrak{h})}{\mathbb{C}^r}$ w/ real coefs.

There is then $R \cong P/P_+^W$ where P_+^W = W-inv't poly w/ $f(0)=0$.

$P \subset S(\mathfrak{h}^*)$ of course

Now, for any root α, the reflection $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ is a linear map that extends to a $\mathbb{C}$-alg aut $S(\mathfrak{h}^*) \to S(\mathfrak{h}^*)$ or a $\mathbb{R}$-alg aut $P \to P$. This preserves P_+^W.

<u>Define</u> $A_\alpha : P \to P$ by $A_\alpha(f) = \dfrac{f - s_\alpha(f)}{\alpha}$ } poly on $\mathfrak{h}$ } elt $\mathfrak{h}^*$, poly of deg 1 on $\mathfrak{h}$

$A_\alpha(f) \in P$ because $f - s_\alpha(f)$ vanishes on the kernel of α and hence is divisible by α.

<u>Theorem</u> (BGG): Let $w \in W$ be given by $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$. where $\alpha_i \in \Delta$ are simple.

① If $\ell(w) < k$ then $A_{\alpha_1} \circ \cdots \circ A_{\alpha_k} = 0$

② If $\ell(w) = k$, then $A_{\alpha_1} \circ \cdots \circ A_{\alpha_k} : R \to R$ depends only on w and not on the word $s_{\alpha_1} \cdots s_{\alpha_k}$.

Using ② we denote the result by $A_w : R \to R$ $w \in W$.

The integration formula is:

<u>Theorem (BGG).</u> Let $\alpha : P \twoheadrightarrow H^*(G/B, \mathbb{R})$ denote the canonical surjection corresp to quotient by W-inv'ts of pos degree. For all $w \in W$ and $f \in P$ we have

$$\langle \alpha(f), X_w \rangle = (A_w f)(0)$$

(Idea: Induction on $\ell(w)$, Study $X_{w'}$ as zero set of section of $\mathcal{L} \to X_w$)

And finally the basis theorem: $\alpha \in \mathfrak{h}^*$ viewed as deg 1 elt P.

<u>Theorem (BGG)</u>. Let $f_{w_0} = \frac{1}{|W|} \prod_{\alpha \in \Phi^+} \alpha$ and for any other $w \in W$ let $f_w = A_{w^{-1}w_0}(f_{w_0})$. Then $\langle \alpha(f_w), X_{w'} \rangle = \delta_{ww'}$.

(Idea: Study commutation properties of A_w & $A_{w'}$.)